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HOMOGENIZATION THEORY OF REGULAR CROSS-PLY LAMINATES

The paper concerns regular cross-ply fibre-reinforced-plastic (CP xFRP) laminates, i.e. a stack of plies of $[0/90]_{nS}$, $n \geq 4$ configuration. Each ply is a UD xFRP composite, i.e., an isotropic hardening plastic reinforced with long monotropic fibres packed unidirectionally in a hexagonal scheme. The plies are identical with respect to their thickness and microstructure. The considerations are limited to stress levels protecting geometrically and physically linear elastic behaviour of the material. The study presents the homogenization theory of a regular CP xFRP laminates. The theory employs respective boundary-value problems put on the representative volume element of the laminate: uniform tension in the *x* direction, uniform tension in the *z* direction, pure shear in the *xz* plane, pure shear in the *xy* plane. The representative volume element is considered in these problems under the following requirements: 1) elastic behaviour of the unhomogenized and homogenized representative volume element must be compatible with behaviour of the whole laminate, 2) the unhomogenized and homogenized representative volume element satisfy the compatibility conditions put on the total stress and displacement states. A concept of homogenization in each boundary-value problem contains the following steps: a) formulation of stress and strain components of the homogenized representative volume element, b) formulation of constitutive equations of elasticity, c) formulation of stress and strain components in each ply of 0° orientation before homogenization, d) formulation of stress and strain components in each ply of 90° orientation before homogenization. The final analytic formulae for effective elasticity constants of the laminate, describing an orthotropic model of the homogenized material, are presented. Based on the exact homogenization theory of UD xFRP composites and the exact stiffness theory of regular CP xFRP laminates presented in this study, the authors have written a computer programme in PASCAL for predicting the effective elasticity constants of these materials. As an example, a regular CP U/E53 laminate of $[0/90]_{nS}$, $n \geq 4$ configuration is considered. The matrix (E53 hardening plastic) is made of Epidian 53 epoxide resin, reinforced with UTS 5631 carbon fibres produced by Tenax Fibers.

Keywords: **regular cross-ply laminate, homogenization, constitutive equations of elasticity, computer-aided algorithm**

TEORIA HOMOGENIZACJI REGULARNYCH LAMINATÓW KRZYŻOWYCH

Praca dotyczy regularnych laminatów krzyżowych z matrycą duroplastyczną, tj. laminatów o konfiguracji warstw $[0/90]_{nS}$, $n \geq 4$. Każda warstwa jest kompozytem wzmocnionym włóknem długim jednokierunkowo, równomiernie, w schemacie heksagonalnym. Warstwy są identyczne w zakresie grubości i mikrostruktury. Rozważania ograniczono do poziomów naprężeń gwarantujących geometrycznie i fizycznie liniowe zachowanie się materiału. Przedstawiono teorię homogenizacji rozpatrywanego typu laminatów. Wyznaczono analityczne formuły określające efektywne stałe sprężystości laminatu modelowanego w wyniku homogenizacji jako ciało ortotropowe. Wykorzystano odpowiednio dobrane zadania brzegowe odniesione do reprezentatywnej objętości laminatu: równomierne rozciąganie w kierunku *x*, równomierne rozciąganie w kierunku *z*, czyste ścinanie w płaszczyźnie *xz*, czyste ścinanie w płaszczyźnie *xy*.

Reprezentatywna objętość laminatu jest analizowana w odpowiednio dobranych zagadnieniach brzegowych przy spełnieniu następujących wymagań: 1) sprężyste zachowanie się reprezentatywnej objętości laminatu przed i po homogenizacji musi być zgodne z zachowaniem całego laminatu, 2) reprezentatywna objętość laminatu przed i po homogenizacji spełnia warunki zgodności nałożone na pełny stan naprężenia lub przemieszczenia. Koncepcja homogenizacji w każdym zagadnieniu brzegowym zawiera następujące etapy: a) sformułowanie stanu naprężenia i odkształcenia reprezentatywnej objętości laminatu po homogenizacji, b) sformułowanie równań konstytutywnych sprężystości, c) sformułowanie stanu naprężenia i odkształcenia w każdej warstwie laminatu o orientacji 0° przed homogenizacją, d) sformułowanie stanu naprężenia i odkształcenia w każdej warstwie laminatu o orientacji 90° przed homogenizacją. Na podstawie ścisłej teorii homogenizacji kompozytów UD xFRP oraz ścisłej teorii sztywności laminatów CP xFRP, sformułowanej w niniejszej pracy, napisano program komputerowy w języku PASCAL do prognozowania wartości efektywnych stałych sprężystości tych materiałów. Jako przykład przedstawiono wyniki obliczeń laminatu regularnego CP U/E53 o konfiguracji $[0/90]_{nS}$, $n \geq 4$. Matrycą jest duroplast E53 wytworzony z żywicy epoksydowej Epidian 53. Wzmocnienie każdej warstwy stanowią włókna UTS 5631 produkowane przez Tenax Fibers.

Słowa kluczowe: **regularne laminaty krzyżowe, homogenizacja, równania konstytutywne sprężystości, komputerowo wspomagany algorytm**

INTRODUCTION

In the classic laminate theory a laminate is modelled as a thin anisotropic plate of constant thickness, subject to the Kirchhoff-Love kinematical hypothesis [1-3]. Each ply is homogenized and modelled as a monotropic solid body. Arbitrary configuration of the plies induces coupling between the membrane and plate states. However, the Kirchhoff-Love kinematical hypothesis is mostly unsatisfied because of relatively low values of the shear moduli. The elastic response of the laminate is better simulated under the Reissner-Mindlin kinematical hypothesis [4]. Reformulation of the classic laminate theory to the first-order shear-deformation theory is very sophisticated [5] but useless from the finite-element-method's point of view. Nowadays, the finite element method is the only effective tool to simulate the elastic behaviour of laminate structures with shear deformation taken into consideration. Preprocessors of CAE systems require values of the effective elasticity constants (EECs) of an anisotropic material modelling a laminate. These values may be identified experimentally, but theoretical analytical prediction of the EECs, using respective homogenization theory, is cheaper, easier and much more important in mechanics of laminates.

On the micromechanics level, a ply (a UD xFRP composite) is modelled as a linearly elastic monotropic continuum with the monotropy axis coinciding the direction of fibres' alignment. The following assumptions are adopted:

- each ply is a two-phase material,
- both constituents, a matrix and a fibre, are homogeneous,
- stresses are restricted to the levels protecting linear behaviour of the constituents,
- there are considered quasi-static isothermal processes in the normal conditions, i.e. the processes belonging to the transition regime under the glass transition temperature,
- a matrix is a chemically hardening plastic made of a crosslinked polymer, modelled as an elastic isotropic material,
- a fibre is modelled as an elastic monotropic material (isotropic, in particular),
- fibres have identical solid circular cross-section; they are rectilinear and embedded uniformly in the matrix, in a hexagonal scheme,
- preparation of the fibres protects perfect bonding of the fibres to the matrix,
- residual stresses resulting from the manufacturing process are neglected.

Each ply (a UD xFRP composite) is described in the $x_1x_2x_3$ - Cartesian coordinate system with x_1 - a monotropy axis, and x_2x_3 - a transverse isotropy plane. The constituents are characterized by the following elasticity constants: E, ν (a Young's modulus, a Poisson's ratio of the matrix), $\overline{E}_1, \overline{E}_2, \overline{\nu}_{32}, \overline{\nu}_{21}, \overline{G}_{12}$ (longitudinal and transverse Young's moduli, Poisson's ratios in respective planes, a shear modulus in the monotropy plane of the fibre). The composite is also described by the fibre volume fraction f. Monotropic continuum modelling the homogenized ply is described by five independent effective elasticity constants (EECs), i.e., E_1, E_2, ν_{32}, ν_{21}, G_{12} (effective longitudinal and transverse Young's moduli, effective Poisson's ratios in respective planes, an effective shear modulus in the monotropy plane). These constants are derived in terms of elasticity constants of the constituents and of the fibre volume fraction from the exact homogenization theory summarized in Ref. [6].

On the mesomechanics level, a regular CP xFRP laminate can be modelled as a homogeneous orthotropic continuum, described in the xyz - Cartesian coordinate system. Axes x, y coincide the lamination directions of respective groups of plies, whereas axis z is perpendicular to the xy midplane.

Standard constitutive equations of linear elasticity of the homogenized (orthotropic) regular CP xFRP laminate have the following well-known form [1-4]

$$\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma} \tag{1}$$

where

$$\begin{aligned} \boldsymbol{\sigma} &= \mathrm{col}\left(\sigma_x, \sigma_y, \sigma_z, \sigma_{yz}, \sigma_{xz}, \sigma_{xy}\right) \\ \boldsymbol{\varepsilon} &= \mathrm{col}\left(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}\right) \end{aligned} \tag{2}$$

are stress and strain vectors in the xyz - system, i.e.: σ_x, σ_y, σ_z - normal stresses, σ_{yz}, σ_{xz}, σ_{xy} - shear stresses, ε_x, ε_y, ε_z - directional strains, γ_{yz}, γ_{xz}, γ_{xy} - shear strains. The elasticity compliance matrix

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \tag{3}$$

contains the following elasticity compliances

$$\begin{aligned} & S_{11} = \frac{1}{E_x},\ S_{22} = \frac{1}{E_y},\ S_{33} = \frac{1}{E_z} \\ & S_{12} = -\frac{\nu_{yx}}{E_x},\ S_{13} = -\frac{\nu_{zx}}{E_x},\ S_{23} = -\frac{\nu_{zy}}{E_y} \\ & S_{44} = \frac{1}{G_{yz}},\ S_{55} = \frac{1}{G_{xz}},\ S_{66} = \frac{1}{G_{xy}} \end{aligned} \tag{4}$$

expressed in terms of the EECs of the homogenized laminate, i.e.

E_x, E_y, E_z - Young's moduli in the x, y, z directions,

ν_{zy}, ν_{zx}, ν_{yx} - Poisson's ratios in respective planes,
G_{yz}, G_{xz}, G_{xy} - shear moduli in respective planes.
For a regular CP xFRP laminate only six EECs take different values, i.e., E_x, E_z, ν_{zx}, ν_{yx}, G_{xz}, G_{xy}.

THE EXACT HOMOGENIZATION THEORY OF REGULAR CP xFRP LAMINATE

There is considered a representative volume element (RVE) cut from the whole laminate at point $A(x,y,0)$, shown in Figure 1 for an exemplary value of $n = 4$. Before homogenization, the RVE is a stack of plies of $[0/90]_{nS}$ configuration ($n \geq 4$) of cubicoidal global geometry. After homogenization, the RVE is a homogeneous orthotropic cubicoid of dimensions $1\times1\times h$ where h is a thickness of the laminate. The orthotropy directions coincide the x, y, z axes, respectively. The RVE is considered in selected boundary-value problems (BVPs) under the following requirements:

- elastic behaviour of the unhomogenized and homogenized RVE must be compatible with behaviour of the whole laminate,
- the unhomogenized and homogenized RVE satisfy the compatibility conditions put on the total stress and displacement states.

In further considerations, a' denotes a quantity related to the plies of 0° arrangement, a'' - a quantity related to the plies of 90° arrangement, and a - a quantity related to the homogenized laminate.

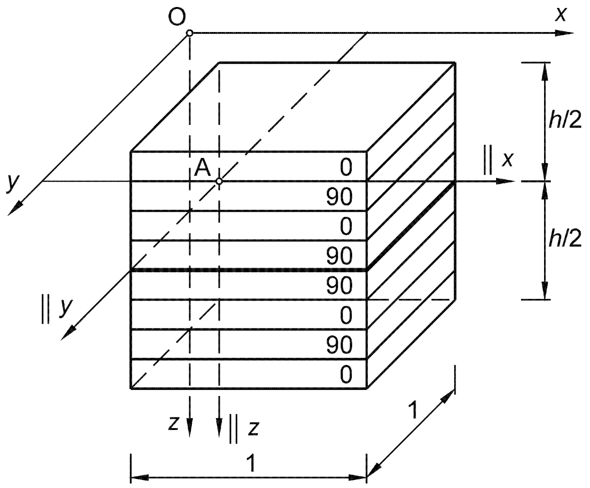

Fig. 1. A representative volume of the laminate for an exemplary value of $n = 4$

Rys. 1. Reprezentatywna objętość laminatu dla przykładowej wartości $n = 4$

A concept of homogenization in each BVP contains the following steps:

a) formulation of stress and strain components of the homogenized RVE,
b) formulation of constitutive equations of elasticity,
c) formulation of stress and strain components in each ply of 0° orientation before homogenization,
d) formulation of stress and strain components in each ply of 90° orientation before homogenization.

The first BVP is uniform tension in the x direction. Following the behaviour of the whole laminate, the RVE walls remain planar. The problem is related to the constrained RVE, namely, the walls parallel to the xz plane are unmovable in the y direction. The shear deformations vanish. The RVE satisfies the following compatibility conditions:

1) compatibility of the stress resultant in the x-direction:

$$\sigma_x'\frac{h}{2} + \sigma_x''\frac{h}{2} = \sigma_x h \tag{5}$$

2) compatibility of the stress resultant in the y-direction:

$$\sigma_y'\frac{h}{2} + \sigma_y''\frac{h}{2} = \sigma_y h \tag{6}$$

3) compatibility of the elongation of the RVE in the z-direction:

$$\varepsilon_z'\frac{h}{2} + \varepsilon_z''\frac{h}{2} = \varepsilon_z h \tag{7}$$

Inserting the results of a), b), c), d) into the compatibility conditions (5), (6), (7) gives first three equations of homogenization, i.e.:

$$\begin{gathered} \frac{E_1 + E_2}{1-\nu_{21}\nu_{12}} = \frac{2E_x}{1-\nu_{yx}^2} \\ \frac{E_1\nu_{12} + E_2\nu_{21}}{1-\nu_{21}\nu_{12}} = \frac{2E_x\nu_{yx}}{1-\nu_{yx}^2} \\ \frac{\nu_{21}(1+\nu_{12}+\nu_{32}) + \nu_{32}}{1-\nu_{21}\nu_{12}} = \frac{2\nu_{zx}}{1-\nu_{yx}} \end{gathered} \tag{8}$$

The second BVP is uniform tension in the z direction. Following the behaviour of the whole laminate, the RVE walls remain planar. The problem is related to the constrained RVE, namely, vertical walls are unmovable in the x and y direction, respectively. The shear deformations vanish. The RVE satisfies the compatibility condition put on the elongation of the RVE in the z direction, i.e.

$$\varepsilon_z'\frac{h}{2} + \varepsilon_z''\frac{h}{2} = \varepsilon_z h \tag{9}$$

Respective insertions into Eq. (9) give fourth equation of homogenization, of the form

$$\frac{1-2\nu_{21}\nu_{12}(1+\nu_{32}) - \nu_{32}^2}{E_2(1-\nu_{21}\nu_{12})} = \frac{1}{E_z} - \frac{2\nu_{zx}^2}{E_x(1-\nu_{yx})} \tag{10}$$

Equations (8), (10) constitute a set of four nonlinear algebraic equations with unknowns E_x, E_z, ν_{zx}, ν_{yx}. One can rewrite them to the form

$$\begin{cases} 2E_x = \left(1-\nu_{yx}^2\right)a \\ 2E_x\nu_{yx} = \left(1-\nu_{yx}^2\right)e \\ 2\nu_{zx} = \left(1-\nu_{yx}\right)c \\ \dfrac{1}{E_z} - \dfrac{2\nu_{zx}^2}{E_x\left(1-\nu_{yx}\right)} = d \end{cases} \tag{11}$$

where

$$a = \frac{E_1+E_2}{1-\nu_{21}\nu_{12}},\ e = \frac{E_1\nu_{12}+E_2\nu_{21}}{1-\nu_{21}\nu_{12}},\ b = \frac{e}{a} = \frac{E_1\nu_{12}+E_2\nu_{21}}{E_1+E_2}$$
$$c = \frac{\nu_{21}\left(1+\nu_{12}+\nu_{32}\right)+\nu_{32}}{1-\nu_{21}\nu_{12}},\ d = \frac{1-2\nu_{21}\nu_{12}\left(1+\nu_{32}\right)-\nu_{32}^2}{E_2\left(1-\nu_{21}\nu_{12}\right)} \tag{12}$$

Solving analytically Eqs. (11) results in first four EECs, i.e.:

$$\begin{aligned} E_x &= \frac{1}{2}a\left(1-b^2\right),\ E_z = \frac{a(1+b)}{a(1+b)d+c^2} \\ \nu_{zx} &= \frac{1}{2}c(1-b),\ \nu_{yx} = b \end{aligned} \tag{13}$$

The third BVP is pure shear in the *xz* plane. Following the behaviour of the whole laminate, shear deformations of the RVE before and after homogenization take the form as shown in Figure 2. Shear angles related to plies of 0° and 90° orientation are different and independent. The horizontal walls of the RVE remain planar. The strength task is related to the unconstrained RVE. The bulk deformations vanish.

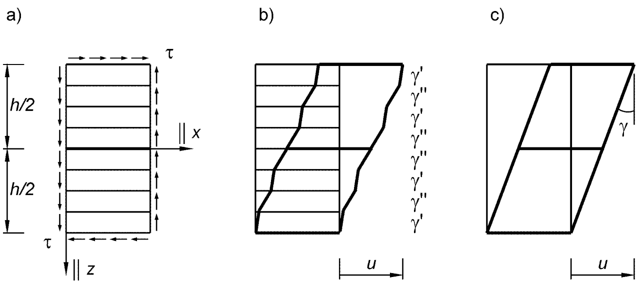

Fig. 2. Pure shear of the RVE in the *xz* plane: a) distribution of shear stresses; b) deformation of the unhomogenized RVE; c) deformation of the homogenized RVE

Rys. 2. Czyste ścinanie reprezentatywnej objętości laminatu w płaszczyźnie *xz*: a) rozkład naprężeń stycznych; b) deformacja reprezentatywnej objętości przed homogenizacją; c) deformacja reprezentatywnej objętości po homogenizacji

The compatibility condition is put on the horizontal shift of the top wall of the RVE, i.e.:

$$\gamma'\frac{h}{2}+\gamma''\frac{h}{2}=\gamma h \tag{14}$$

Inserting respective formulae for shear deformations into Eq. (14) yields the equation

$$\frac{1}{G_{12}}+\frac{1}{G_{23}}=\frac{2}{G_{xz}} \tag{15}$$

giving a shear modulus in the *xz* plane, i.e. the fifth EEC

$$G_{xz}=\frac{2G_{12}G_{23}}{G_{12}+G_{23}} \tag{16}$$

The last BVP is pure shear in the *xy* plane. In the horizontal plane, both groups of the plies are described by the same Kirchhoff's modulus G_{12}. Shear deformations of the unconstrained RVE before and after homogenization are identical - a cubicoid metamorphoses into a parallelepiped. It results in

$$G_{xy}=G_{12} \tag{17}$$

NUMERICAL EXAMPLE

Based on the exact homogenization theory of UD xFRP composites [6] and the exact stiffness theory of regular CP xFRP laminates presented in this study, the authors have written a computer programme in PASCAL for predicting the ECCs of these materials.

As an example, a regular CP U/E53 laminate of $[0/90]_{nS}$, $n \geq 4$ configuration is considered. The matrix (E53 hardening plastic) is made of Epidian 53 epoxide resin, reinforced with UTS 5631 carbon fibres produced by Tenax Fibers. The elasticity constants of the constituents equal [7]

$$\begin{aligned} &E = 3.1\ \text{GPa},\ \nu = 0.42 \\ &\overline{E}_1 = 234\ \text{GPa},\ \overline{E}_2 = 6.6\ \text{GPa} \\ &\overline{\nu}_{32} = 0.36,\ \overline{\nu}_{21} = 0.11,\ \overline{G}_{12} = 10.6\ \text{GPa} \end{aligned}$$

The fibre volume fraction equals $f = 0.50$.

The EECs of a single ply, calculated according to the homogenization theory presented in Ref. [6], equal

$$\begin{aligned} &E_1 = 118.6\ \text{GPa},\ E_2 = 5.4\ \text{GPa} \\ &\nu_{32} = 0.54,\ \nu_{21} = 0.27,\ G_{12} = 2.6\ \text{GPa} \end{aligned}$$

The EECs of the CP U/E53 laminate, calculated according to the homogenization theory presented in this study, equal

$$\begin{aligned} &E_x = E_y = 62.2\ \text{GPa},\ E_z = 7.3\ \text{GPa} \\ &\nu_{zx} = \nu_{zy} = 0.47,\ \nu_{yx} = \nu_{xy} = 0.02 \\ &G_{xz} = G_{yz} = 2.1\ \text{GPa},\ G_{xy} = 2.6\ \text{GPa} \end{aligned}$$

CONCLUSIONS

The paper concerns regular CP xFRP laminates, i.e., a stack of plies of $[0/90]_{nS}$, $n \geq 4$ configuration. Each ply is a UD xFRP composite, i.e. an isotropic hardening plastic reinforced with long monotropic fibres packed unidirectionally in a hexagonal scheme. The plies are identical with respect to their thickness and microstructure. The considerations are limited to stress levels protecting geometrically and physically linear elastic behaviour of the material.

The exact homogenization theory of a regular CP xFRP laminate has been developed. Effective elasticity constants of the homogenized orthotropic laminate have been derived analytically from respective BVPs related to the representative volume element of the laminated.

A computer-aided algorithm for calculation of the effective elasticity constants of regular CP xFRP laminates has been formulated, programmed in PASCAL and applied for a number of materials. A selected example is presented in the study. Correctness, high accuracy and practical usability of the algorithm has been confirmed.

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