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SYMMETRIC, BALANCED CROSS-PLY AND DIAGONAL-PLY LAMINATES. GLOBAL ELASTIC PROPERTIES AND INTERNAL STRESSES

In the work, research has been taken up on the problem of lead elimination from presently very popular technical Journal Composites [1] presented a paper by T. Niezgoda and M. Klasztorny, giving predicted values of elastic constants for a special case of balanced laminates. The present paper is influenced by the idea from the mentioned work, where the described case can be obtained without the use full methods of the theory of lamination, employing instead simple engineering techniques, which in turn can be solved symbolically and can bring a better understanding of the problems of lamination. The relations given here show the step by step procedure of obtaining the necessary elasticity relations and stress values.

The main aim in presenting this paper is twofold. Primarily the employed methods are easy to understand and useful in the cases of first contact with the theory of lamination. Secondly, the method of obtaining presented results for diagonal-ply laminates, using a simple rotation of coordinates, is unique and to the Authors knowledge has not been published up to now. There are also other reasons for presenting this paper for not only there are insufficient data to follow the original procedure but also there is an error in the called literature, which now here is presented as [2]. Unfortunately this paper obtained by personal communication does not help much.

Another item in this presentation is giving the exact, useful values of material constants for a diagonal laminate, together with values of acting stresses for this case, which may be of further use. As a result the present paper gives a detailed explanation of the procedure used in the homogenization process, shows application of this method to another commonly used composite and presents useful values of internal stresses, which decide on strength of the structure.

Keywords: **cross-ply laminate, constitutive equations, global constants of elasticity, conditions for cohesion and homogenization, internal stresses**

ZRÓWNOWAŻONE, SYMETRYCZNE LAMINATY KRZYŻOWE I LAMINATY DIAGONALNE. WŁASNOŚCI SPRĘŻYSTE I NAPRĘŻENIA WEWNĘTRZNE

Czasopismo Kompozyty [1] przedstawiło pracę T. Niezgody i M. Klasztornego, podającą przewidywane wartości stałych sprężystości specjalnego typu zrównoważonych laminatów. Praca obecna jest wywołana pomysłem pochodzącym z wymienionej publikacji, z którego wynika, że w specjalnych przypadkach możliwe jest przewidywanie własności laminatu, używając prostych metod inżynierskich, bez uciekania się do pełnej teorii laminowania.

Przedstawienie niniejszej pracy ma dwa cele. Po pierwsze, zdaniem Autorów, zaprezentowana metoda jest bardzo przydatna, szczególnie w przypadku pierwszego kontaktu z teorią laminowania kompozytów i może być z pożytkiem stosowana podczas początkowych zajęć dydaktycznych z tych zagadnień. Przedstawiono krok po kroku proces tworzenia zależności opisujących wartości stałych sprężystości laminatu oraz wartości poszczególnych naprężeń. Celem drugim jest pokazanie wyjątkowo prostej metody uzyskiwania wyników w opisie laminatów diagonalnych, polegającej na prostym obrocie układu współrzędnych, nieznanej dotychczas zdaniem Autorów, z innych publikacji.

Innym powodem opracowania obecnej pracy jest to, że ani z publikacji [1], ani z publikacji [2], błędnie cytowanej w wymienionej pracy przez obu autorów, a uzyskanej drogą osobistego przekazu, nie wynika jasno metoda postępowania, prowadząca do przedstawionych wyników. W efekcie tych stwierdzeń powstała niniejsza praca, pokazująca pełną drogę postępowania w badanych prostych przypadkach, co być może ułatwi przekazanie istoty aspektów teorii laminowania w prostej formie. Niezależnie od tego rozważono inny, równie prosty przypadek szczególny często występującego w praktyce laminatu, a także podano zależności umożliwiające określenie wartości naprężeń wewnętrznych, decydujących o wytrzymałości laminatu. Wreszcie w przypadkach prób oceny wpływu poszczególnych składników kompozytu czy laminatu na własności powstałego materiału zależności podane w formie symbolicznej okazują się bardzo przydatne.

Słowa kluczowe: **laminaty krzyżowe, równania konstytutywne, globalne stałe sprężystości, warunki spójności i homogenizacji, naprężenia wewnętrzne**

BASIC ASSUMPTIONS

This paper deals with a special case of lamination theory, presenting results of homogenization for a balanced cross-ply laminate of a specific structure. The laminate consists of an even number of monotropic layers, stacked alternatively in perpendicular directions in the laminate plane, as shown in Figure 1.

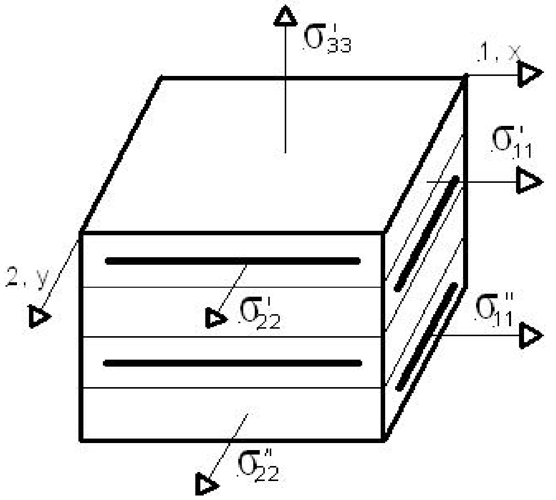

Fig. 1. Structure of the laminate

Rys. 1. Układ kompozytu

Properties of such a laminate can be obtained using the general theory of lamination as given in [3] or [4] but also by a much more direct method as described bellow.

MATERIAL PROPERTIES

For a linear theory of anisotropic elasticity properties of a composite monotropic layer can be given by a set of engineering elastic constants

$$E_{11},\, E_{22}=E_{33},\, \nu_{32},\, \nu_{21},\, G_{31}=G_{12},\, G_{23}=\frac{E_{22}}{2(1+\nu_{32})} \qquad (1)$$

and a relation describing a symmetry

$$\frac{\nu_{12}}{E_{22}}=\frac{\nu_{13}}{E_{22}}=\frac{\nu_{21}}{E_{11}} \qquad (2)$$

For a symmetric, balanced cross-ply laminate resulting principal directions of elasticity can be predicted as coinciding with the layers directions 1, 2, and 3. To differentiate the global elastic properties instead of 1, 2, 3 the subscripts x, y, z will be used. Additionally the monotropic layer with the symmetry axis in the direction of 1 will be marked with $'$ while the one in the perpendicular direction 2 with $''$. For such a case the global elastic properties of the laminate will be described by

$$E_{xx}=E_{yy},\, \nu_{xy},\, \nu_{yz}=\nu_{zx},\, G_{xz}=G_{yz},\, G_{xy} \qquad (3)$$

and the symmetry relations

$$\frac{\nu_{xy}}{E_{yy}}=\frac{\nu_{yx}}{E_{xx}},\ \frac{\nu_{zx}}{E_{xx}}=\frac{\nu_{xz}}{E_{zz}} \qquad (4)$$

The current problem is to determine all the elastic constant values given by (3). Constitutive relations of the laminate properties can be conveniently described using engineering constants as:

$$\begin{aligned}
\varepsilon_{xx}&=\frac{\sigma_{xx}}{E_{xx}}-\frac{\nu_{yx}}{E_{xx}}\sigma_{yy}-\frac{\nu_{zx}}{E_{xx}}\sigma_{zz},\ \varepsilon_{xy}=\frac{\sigma_{xy}}{2G_{xy}}\\
\varepsilon_{yy}&=\frac{\sigma_{yy}}{E_{yy}}-\frac{\nu_{zy}}{E_{yy}}\sigma_{zz}-\frac{\nu_{xy}}{E_{yy}}\sigma_{xx},\ \varepsilon_{yz}=\frac{\sigma_{yz}}{2G_{yz}}\\
\varepsilon_{zz}&=\frac{\sigma_{zz}}{E_{zz}}-\frac{\nu_{xz}}{E_{zz}}\sigma_{xx}-\frac{\nu_{yz}}{E_{zz}}\sigma_{yy},\ \varepsilon_{zx}=\frac{\sigma_{zx}}{2G_{zx}}
\end{aligned} \qquad (5)$$

Replacing the subscripts x, y, z by 1, 2, 3 will lead to constitutive relations for a single layer, the direction of which can be differentiated by $'$ and $''$.

LOADS IN THE LAMINATE PLANE

In this case the following conditions of equilibrium and compatibility are to be met:

$$\begin{aligned}
&\sigma'_{11}+\sigma''_{11}=2\sigma_{xx},\ \sigma'_{22}+\sigma''_{22}=2\sigma_{yy},\ \sigma'_{33}=\sigma''_{33}=\sigma_{zz}=0\\
&\varepsilon'_{11}=\varepsilon''_{11}=\varepsilon_{xx},\ \varepsilon'_{22}=\varepsilon''_{22}=\varepsilon_{yy},\ \varepsilon'_{33}+\varepsilon''_{33}=2\varepsilon_{zz}\\
&\sigma'_{12}=\sigma'_{23}=\sigma'_{31}=0,\ \sigma''_{12}=\sigma''_{23}=\sigma''_{31}=0,\ \sigma_{xy}=\sigma_{yz}=\sigma_{zx}=0
\end{aligned} \qquad (6)$$

These conditions lead to an equation in the direction of e.g. 1

$$\frac{\sigma'_{11}}{E_{11}}-\frac{\nu_{21}}{E_{11}}\sigma'_{22}=\frac{\sigma''_{11}}{E_{22}}-\frac{\nu_{12}}{E_{22}}\sigma''_{22} \qquad (7)$$

Using the equilibrium condition in the direction of 1 allows to transform (7) into

$$\frac{\sigma'_{11}}{E_{11}}-\frac{\nu_{21}}{E_{11}}=-\frac{\sigma'_{11}}{E_{22}}+\frac{\nu_{12}}{E_{22}}\sigma'_{22}+\frac{2}{E_{22}}(\sigma_{xx}-\nu_{12}\sigma_{yy})$$

and finally to an equation

$$(E_{11}+E_{22})\sigma'_{11}-2\nu_{12}E_{11}\sigma'_{22}=2E_{11}(\sigma_{xx}-\nu_{12}\sigma_{yy}) \qquad (8)$$

In a similar way, for the direction 2 can be formed an equation

$$-2\nu_{21}E_{22}\sigma'_{11}+(E_{11}+E_{22})\sigma'_{22}=2E_{22}(\sigma_{yy}-\nu_{21}\sigma_{xx}) \qquad (9)$$

The equations (8) and (9) allow to find necessary I some cases internal stresses in both layers, using the following determinants

$$\Delta = E_{11}^2 (1-\nu_{12}^2) + 2E_{11}E_{22}(1-\nu_{12}\nu_{21}) + E_{22}^2(1-\nu_{21}^2)$$
$$\Delta_{11} = 2E_{11}\left[(E_{11}+E_{22})(1-2\nu_{12}\nu_{21})\right]\sigma_{xx} - \nu_{12}(E_{11}-E_{22})\sigma_{yy}$$
$$\Delta_{22} = -2E_{22}\nu_{21}(E_{11}-E_{22})\sigma_{xx} - \left[E_{11}(1-2\nu_{12}\nu_{21}) + E_{22}\right]\sigma_{yy} \tag{10}$$

can be presented as

$$\sigma'_{11} = \frac{\Delta_{11}}{\Delta},\ \sigma''_{11} = 2\sigma_{xx} - \frac{\Delta_{11}}{\Delta}$$
$$\sigma'_{22} = \frac{\Delta_{22}}{\Delta},\ \sigma''_{22} = 2\sigma_{yy} - \frac{\Delta_{22}}{\Delta} \tag{11}$$

These results are sufficient to calculate two global elastic constants acting in the laminate plane. Comparison of strains in the direction 1 gives

$$\frac{\sigma'_{11}}{E_{11}} - \frac{\nu_{21}}{E_{11}}\sigma'_{22} = \frac{\sigma_{xx}}{E_{xx}} - \frac{\nu_{yx}}{E_{xx}}\sigma_{yy} \tag{12}$$

which allows for further calculations. Equating successively $\sigma_{yy} = 0$ and $\sigma_{xx} = 0$ and using the above comparison the following can be found

$$\frac{1}{E_{xx}} = 2\frac{(E_{11}+E_{22})(1-\nu_{12}\nu_{21})}{\Delta}$$
$$\frac{\nu_{yx}}{E_{xx}} = 2\frac{(E_{11}\nu_{12}+E_{22}\nu_{21})(1-\nu_{12}\nu_{21})}{\Delta} \tag{13}$$

Further still the results obtained can also be useful for determination of strains, caused by such loads in the direction of 3. In general this global strain can be calculated as

$$2\varepsilon_{33} = -\frac{1}{E_{22}}(\nu_{12}\sigma'_{11} + \nu_{32}\sigma'_{22} + \nu_{32}\sigma''_{11} + \nu_{12}\sigma''_{22})$$

It is sufficient to take into account only one of the stresses acting in the laminate, e.g. σ_{xx}. In such a case, due to inserting $\sigma_{yy} = 0$ into (11), the above relation reduces to

$$2\varepsilon_{33} = -\frac{1}{E_{zz}}\left[(\nu_{12}-\nu_{32})(\sigma'_{11}-\sigma'_{22}) + 2\nu_{32}\sigma_{xx}\right] \tag{14}$$

Performing lengthy but necessary calculations an expression is found

$$\frac{\varepsilon_{33}\Delta}{\sigma_{xx}} = -(E_{11}+E_{22})\left[\nu_{21}(1+\nu_{12}) + \nu_{32}(1-\nu_{21})\right] +$$
$$+ 2\nu_{21}E_{11}(\nu_{12}-\nu_{32})(1+\nu_{12})$$

from the above expression a global constant is found

$$\frac{\nu_{zx}}{E_{xx}} = \frac{E_{11}+E_{22}}{\Delta}\left[\nu_{21}(1+\nu_{12}) + \nu_{32}(1-\nu_{21})\right] +$$
$$+ 2\nu_{21}E_{11}(\nu_{12}-\nu_{32})(1+\nu_{12}) \tag{15}$$

for which the symmetry relation (4) is valid as well. Internal stresses as previously are given by (11).

LOADS PERPENDICULAR TO THE LAMINATE PLANE

For a case, where the only existing global stress is σ_{zz}, following conditions for the laminate apply

$$\sigma'_{33} = \sigma''_{33} = \sigma_{zz},\ \sigma'_{11} + \sigma''_{11} = 0,\ \sigma'_{22} + \sigma''_{22} = 0$$
$$\varepsilon'_{33} = \varepsilon''_{33} = \varepsilon_{zz},\ \varepsilon'_{11} = \varepsilon''_{11} = \varepsilon_{xx},\ \varepsilon'_{22} = \varepsilon''_{22} = \varepsilon_{xx} \tag{16}$$
$$\sigma'_{12} = \sigma'_{23} = \sigma'_{31} = 0,\ \sigma''_{12} = \sigma''_{23} = \sigma''_{31} = 0$$

Additionally, because of symmetry and identical properties of the constituent layers, it can be expected that

$$\sigma'_{11} = \sigma''_{22},\ \sigma'_{22} = \sigma''_{11},\ \varepsilon_{xx} = \varepsilon_{yy} \tag{17}$$

Equating strains in the layers e.g. in the 1 direction

$$\varepsilon'_{11} = \frac{\sigma'_{11}}{E_{11}} - \frac{\nu_{12}}{E_{22}}\sigma'_{22} - \frac{\nu_{32}}{E_{22}}\sigma_{zz}$$
$$\varepsilon''_{11} = \frac{\sigma''_{11}}{E_{22}} - \frac{\nu_{12}}{E_{22}}\sigma''_{22} - \frac{\nu_{32}}{E_{22}}\sigma_{zz}$$

it can be found that a relation exists

$$\sigma'_{11}(E_{11}+E_{22}) = 2\nu_{12}E_{11}\sigma'_{22} \tag{18}$$

From the (17) relations and using the compatibility of strains from (16) follows that

$$\frac{\sigma'_{22}}{E_{22}}(1+\nu_{12}) - \frac{\nu_{32}}{E_{22}}\sigma_{zz} = -\frac{\nu_{xz}}{E_{zz}}\sigma_{zz} = -\frac{\nu_{zx}}{E_{xx}}\sigma_{zz}$$

and this determines the value of σ'_{22} as

$$\sigma'_{22} = \frac{E_{22}}{1+\nu_{12}}\left(\frac{\nu_{32}}{E_{22}} - \frac{\nu_{zx}}{E_{xx}}\right)\sigma_{zz} \tag{19}$$

Finally, equating $\varepsilon_{zz} = \varepsilon'_{33}$, replacing σ'_{11} adequately and using (19) a relation is found

$$\varepsilon_{zz} = \left[\frac{1}{E_{22}} + \frac{\nu_{12}-\nu_{32}}{1+\nu_{12}}\left(\frac{\nu_{32}}{E_{22}} - \frac{\nu_{zx}}{E_{xx}}\right)\right]\sigma_{zz}$$

from which directly follows the value of the global elastic constant

$$\frac{1}{E_{zz}} = \frac{1}{E_{22}} + \frac{\nu_{12} - \nu_{32}}{1 + \nu_{12}} \left(\frac{\nu_{32}}{E_{22}} - \frac{\nu_{zx}}{E_{xx}} \right) \qquad (20)$$

where all the necessary values included in this relation are given in (15).

In this way all global constants, responsible for normal stresses and strains have been established and presented symbolically, the remaining global constants are acting in shear.

PURE SHEAR IN THE 1-2 PLANE

In the 1-2 planes of both constituents acts the shear modulus G_{12}. As the directions 1,2 and x,y coincide, the global shear modulus

$$G_{xy} = G_{12} \qquad (21)$$

The remaining conditions for stresses and strains in this case are

$$\begin{aligned} &\sigma_{xx} = \sigma'_{11} = \sigma''_{11} = 0,\ \sigma_{yy} = \sigma'_{22} = \sigma''_{22} = 0,\ \sigma_{yz} = \sigma_{zx} = 0 \\ &\varepsilon_{xx} = \varepsilon'_{11} = \varepsilon''_{11} = 0,\ \varepsilon_{yy} = \varepsilon'_{22} = \varepsilon''_{22} = 0,\ \varepsilon_{yz} = \varepsilon_{zx} = 0 \end{aligned} \qquad (22)$$

At this stage the only remaining unknown engineering constant is the $G_{yz} = G_{zx}$ modulus.

SHEAR IN THE DIRECTIONS 1-3 AND 2-3

In the layer $'$ in the plane 1-3 exists a shear modulus G_{12} while in the 2-3 plane acts the modulus G_{23}. In the layer $''$ the properties are reversed and in the plane 1-3 acts the modulus G_{23} while in the plane 2-3 acts the modulus G_{13}. The conditions of equilibrium and compatibility lead to a set of dependencies

$$\begin{aligned} &\sigma_{zx} = \sigma'_{31} = \sigma''_{31},\ \sigma_{yz} = \sigma'_{23} = \sigma''_{23},\ \varepsilon_{xz} = \varepsilon_{yz} \\ &\varepsilon'_{13} + \varepsilon''_{13} = 2\varepsilon_{xz},\ \varepsilon'_{23} + \varepsilon''_{23} = 2\varepsilon_{yz},\ G_{xz} = G_{yz} \\ &\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0 \end{aligned} \qquad (23)$$

These conditions allow to form two relations

$$2\frac{\sigma_{xz}}{2G_{xz}} = \frac{\sigma'_{13}}{2G_{12}} + \frac{\sigma''_{13}}{2G_{23}},\ 2\frac{\sigma_{yz}}{2G_{yz}} = \frac{\sigma'_{23}}{2G_{23}} + \frac{\sigma''_{23}}{2G_{12}}$$

From the above relations the resulting value of the shear modules can be found

$$G_{xz} = G_{yz} = \frac{2G_{12}G_{23}}{G_{12} + G_{23}} \qquad (24)$$

Internal stresses can also, if need be, presented in a symbolic form but to save space are not reproduced here. They can easily be evaluated using the relations (11), (13), (16), (20) and (24).

SYMMETRIC, DIAGONAL-PLY, BALANCED LAMINATE

If the coordinate system used in the previous sections would be rotated in the laminate plane by 45 degrees, the laminate in the new axes u, v would be a diagonal-ply laminate. The situation is presented in Figure 2.

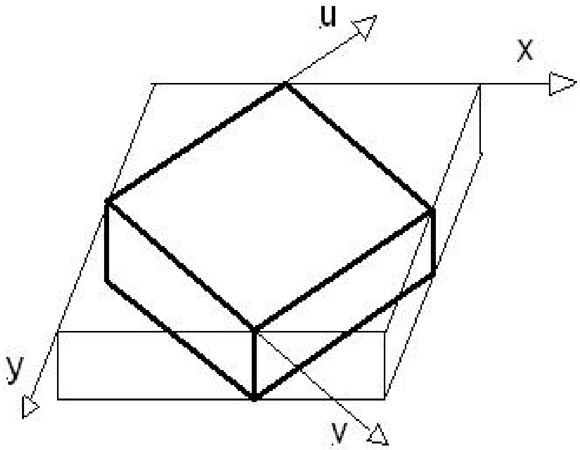

Fig. 2. Rotation of axes
Rys. 2. Obrót osi

There is no problem in transforming the values of stresses, strains and elastic compliance constants between the two coordinate systems, namely x, y, z and u, v, w. For this case the stresses follow the rules listed bellow

$$\begin{aligned} &\sigma_{xx} = \sigma_{yy} = 0.5\ \ \sigma_{uu} + \sigma_{vv}\ \ ,\ \sigma_{xy} = -0.5\ \ \sigma_{uu} - \sigma_{vv} \\ &\sigma_{yz} = 0.5\sqrt{2}\ \ \sigma_{vw} - \sigma_{wu}\ \ ,\ \sigma_{zx} = 0.5\sqrt{2}\ \ \sigma_{vw} + \sigma_{wu} \\ &\sigma_{zz} = \sigma_{ww} \end{aligned} \qquad (25)$$

It is also possible to obtain symbolic representation of the above relations which in term in comparison with the previous one appear somewhat complicated. Transformation of global compliances is also straightforward and according to e.g. [4] can be presented as

$$\begin{aligned} &\frac{1}{E_{uu}} = \frac{1}{E_{vv}} = 0.5\frac{1}{E_{xx}} + 0.25\left(\frac{1}{G_{xy}} - 2\frac{\nu_{yx}}{E_{xx}}\right),\ \frac{1}{E_{ww}} = \frac{1}{E_{zz}} \\ &\frac{\nu_{vu}}{E_{uu}} = -\left(\frac{1-\nu_{yx}}{2E_{xx}} - 0.25\frac{1}{G_{xy}}\right),\ \frac{\nu_{wu}}{E_{uu}} = \frac{\nu_{zx}}{E_{xx}} \\ &\frac{1}{G_{vw}} = \frac{1}{G_{xz}},\ \frac{1}{G_{wu}} = \frac{1}{G_{xy}},\ \frac{1}{G_{uv}} = \frac{2\ \ 1+\nu_{yx}}{E_{xx}} \end{aligned}$$

The above relations (26) can be also presented in a symbolic form but the complication of the result rather rules it out.

CONCLUSIONS

The presented method of calculation of the values of the compliances and internal stresses in two special kinds of laminate may be beneficial in explaining the simple rules of the theory of lamination without the more or less complicated mathematical background of the full theory. On the other hand the possibility of symbolic presentation of the results can be very useful when attempting to establish relations between properties of particular components of the composite and the global resulting properties. This was the main purpose for preparation of this paper.

The obtained results were confronted with full agreement with the results of the classical theory of lamination, given in e.g. [2] or [3] and also the numeric results from [1] are confirmed using the relations presented in this paper.

REFERENCES

[1] Niezgoda T., Klasztorny M., Homogenization theory of regular cross-ply laminates, Kompozyty (Composites) 2009, 9, 2, 154-159.

[2] Klasztorny M., Piekarski R., Advances in the exact stiffness theory of UDxFRP composites, Prace Naukowe, Mechanika, z. 219, OWPW, Warszawa 2007, 251-261.

[3] Datoo H.M., Mechanics of fibrous composites, Elsevier 1991.

[4] Wilczyński A.P., Polimerowe kompozyty włókniste, WNT, Warszawa 1996.

